LETTER TO THE EDITOR

Alternating steady state in one-dimensional flocking

O J O'Loan and M R Evans

Department of Physics and Astronomy, University of Edinburgh, Mayfield Road, Edinburgh EH9 3JZ, Scotland

Abstract. We study flocking in one dimension, introducing a lattice model in which particles can move either left or right. We find that the model exhibits a continuous nonequilibrium phase transition from a condensed phase, in which a single 'flock' contains a finite fraction of the particles, to a homogeneous phase; we study the transition using numerical finite-size scaling. Surprisingly, in the condensed phase the steady state is alternating, with the mean direction of motion of particles reversing stochastically on a timescale proportional to the logarithm of the system size. We present a simple argument to explain this logarithmic dependence. We argue that the reversals are essential to the survival of the condensate. Thus, the discrete directional symmetry is not spontaneously broken.

PACS numbers: 05.70.Fh, 64.60.Cn, 64.60.Ht

Flocking—the collective motion of a large number of self-propelled entities—is a behaviour exhibited by many living beings such as birds, fish and bacteria. Generically, flocks are driven, non-equilibrium systems with many degrees of freedom and as such they have recently attracted much attention in the physics community [1, 2, 3, 4, 5, 6, 7].

Vicsek et al. [1] introduced a simple microscopic model in dimensions $d \geq 2$ in which particles move with a constant speed; they interact only by tending to align with their neighbours. Simulations [2] and a continuum theory [3] showed the existence of a low-noise ordered phase in which the mean velocity of the particles is non-zero, i.e., a phase in which the rotational symmetry of the model is spontaneously broken.

Flocking in one dimension (1d) is less relevant to biological systems than in higher dimensions, but is nevertheless interesting from a fundamental viewpoint. The models studied to date in $d \geq 2$ possess a continuous (rotational) symmetry. The 1d case is necessarily different—since particles are constrained to move either left or right on a line, the underlying directional symmetry is discrete. Czirók et al. [5] have introduced an off-lattice model of 1d flocking in which particles with continuous velocity variables move on a line. Particles tend to move in the same direction as their neighbours but 'errors' are made due to the presence of noise. Simulations and a continuum theory indicate that a continuous phase transition occurs as the noise or particle density is varied [5], the ordered phase being characterised by the presence of a large 'flock'.

When a phase transition occurs from a disordered to an ordered phase, it is usually accompanied by spontaneous symmetry breaking; in the ordered phase, the spontaneously broken symmetry is accompanied by ergodicity breaking in the thermodynamic limit. In an ordered system with discrete symmetry, such as the Ising model in $d \geq 2$, one expects the 'flip time'—the time the system takes to move from one symmetry-related state to another—to diverge exponentially with system size.

In this work we study a simple lattice model of flocking in 1d. We show, using finite-size scaling, that the model has a continuous phase transition from a homogeneous to a condensed phase. However, the condensed phase is not symmetry-broken, but *alternating*—we find that the mean particle velocity alternates its sign on a timescale which grows only logarithmically with system size; moreover, these reversals are essential to the maintenance of the order.

We now define the model that we study. We consider N particles on a periodic 1d lattice of L sites. The particle density is $\rho = N/L$. Each particle, labelled by μ, has a position $x_\mu \in \{1, 2, \ldots, L\}$ and a velocity (or direction) $v_\mu = \pm 1$. To update the system, a particle is chosen at random. Its velocity v_μ is flipped with probability W_μ and then $x_\mu \to x_\mu + v_\mu$. The flip probability W_μ is given by

$$W_\mu = \left[1 - (1 - 2\eta) v_\mu U(x_\mu)\right]/2, \tag{1}$$

where $U(y)$ is the velocity of the majority of the particles (including particle μ) at site y and its two nearest neighbours; we take $U = 0$ when there is no majority. In brief, particles acquire the velocity of the majority of their neighbours with probability $1 - \eta$.

While our lattice model is somewhat simpler than the off-lattice model of reference [5], it is conceptually similar. However, whereas the off-lattice model has only one source of noise (a random perturbation added to the velocity of a particle at each time step), the noise in our model has two distinct sources. The first is due to the flipping of particle velocities and its strength is parameterised by η; the second is due to the random sequential dynamics of the model. Thus, unlike the off-lattice model, our model does not become deterministic in the limit $\eta \to 0$.

Before showing that the model exhibits a continuous phase transition, we first discuss the qualitative behaviour. When η is large, the system has a fairly homogeneous steady state, illustrated in figure 1(a) by a space-time plot for $\eta = 0.2$. However, when η is small, 'condensation' occurs—a large fraction of the particles are contained in a single large flock, which we refer to as the *condensate*. A space-time plot of the steady state for $\eta = 0.02$ is shown in figure 1(b). The direction of motion of the condensate reverses at fairly regular intervals. These reversals are sometimes, but not always, triggered by a collision with a smaller flock travelling in the opposite direction. Just after the condensate has reversed its direction, it is very dense. It then gradually spreads out, becoming more diffuse at the front than at the back, before reversing again. The steady state clearly has no time-reversal symmetry and is *alternating*, by which we mean that the system reverses its mean velocity at stochastic intervals. A quantity measuring the spatial order in the system, such as the variance of the number of particles per site,

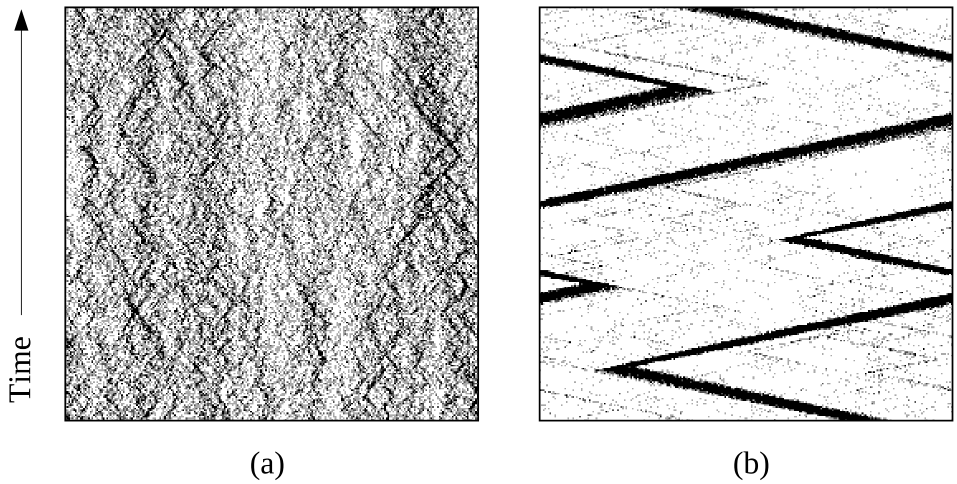


Figure 1. Space-time plots of steady-state systems with (a) $\eta = 0.2$ and (b) $\eta = 0.02$. In both cases, $\rho = 1$ and $L = 1000$. A darker grey level indicates a higher particle density. There is one timestep between each snapshot in (a) and 5 in (b).

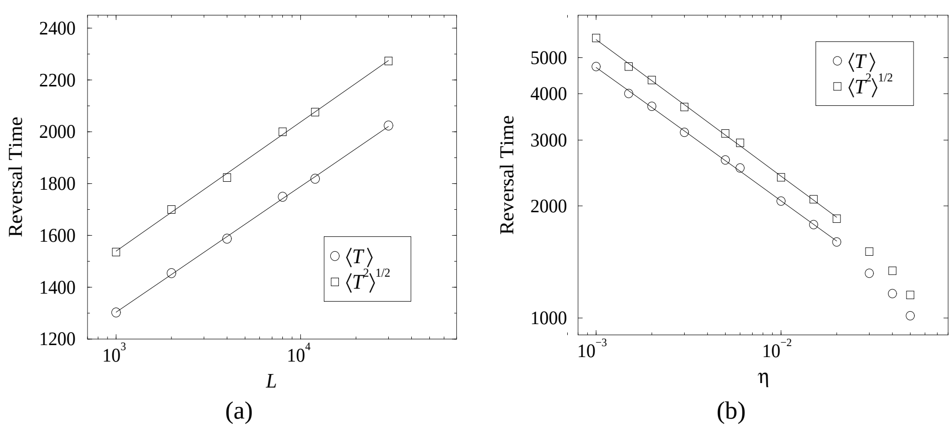


Figure 2. (a) Log-linear plot of $\langle T \rangle$ and $\langle T^2 \rangle^{1/2}$ as a function of L for $\eta = 0.02$ and $\rho = 1$. The straight lines are fits to the data. (b) Log-log plot of $\langle T \rangle$ and $\langle T^2 \rangle^{1/2}$ as a function of η for $L = 4000$ and $\rho = 1$. The solid lines are fits to the data for $\eta \leq 0.02$ with exponents of -0.358 ± 0.01 and -0.366 ± 0.01 respectively.

decreases gradually as the condensate spreads, then increases suddenly as the condensate 'flips', before decaying again.

If a condensate were never to reverse its direction of motion, it would eventually become so diffuse that it would cease to exist—the alternating character of the steady state is essential for the existence of the condensate. The condensate has no preferred direction of motion and we conclude that directional symmetry is not broken.

We now discuss the behaviour of the condensate in more detail. Let $p(T)$ be the probability that the condensate reverses its direction (i.e. that the mean particle velocity changes sign) a time T after the previous reversal. Figure 2(a) shows a log-linear plot of $\langle T \rangle$ and $\langle T^2 \rangle^{1/2}$ as a function of system size L for fixed $\eta = 0.02$. Both are proportional

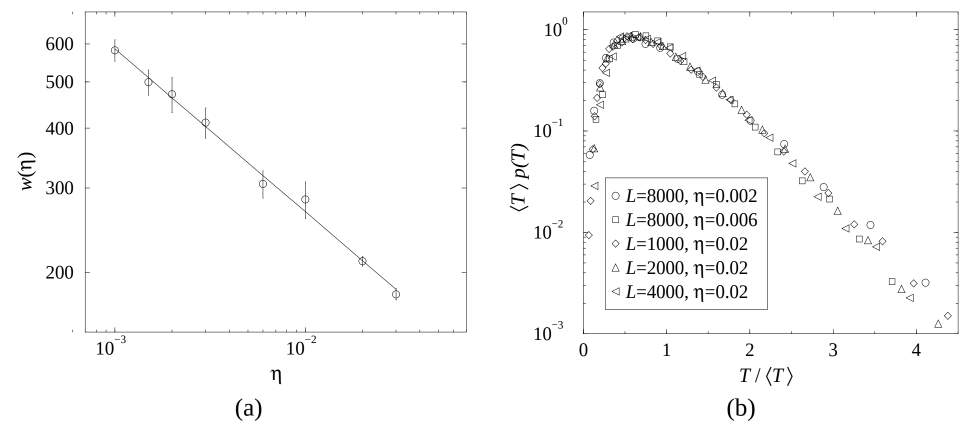


Figure 3. (a) Log-log plot of $w(\eta)$ for $\rho = 1$. The solid line is a fit to the data. (b) Linear-log plot of the scaled reversal time distribution $\langle T\rangle p(T)$ as a function of scaled time $T/\langle T\rangle$ for $\rho = 1$ and various values of L and η.

to $\log L$. Thus, while no spontaneous symmetry breaking occurs, the mean time between reversals does diverge weakly in the thermodynamic limit, as does the variance.

Figure 2(b) shows a log-log plot of $\langle T\rangle$ and $\langle T^2\rangle^{1/2}$ as a function of η for fixed $L = 4000$. For η less than about 0.02, both $\langle T\rangle$ and $\langle T^2\rangle^{1/2}$ diverge as power laws in η. The data suggests that the exponent, which we call λ, is the same in both cases with a value of -0.36 ± 0.01. We have seen in figure 2(a) that $\langle T\rangle \propto \log L$ for fixed η and L large; we denote the constant of proportionality by $w(\eta)$. Figure 3(a) shows a log-log plot of $w(\eta)$ and a least-squares fit to the data gives an estimate of -0.34 ± 0.02 for the slope; it seems likely that this is equal to λ and so we conclude that $\langle T\rangle \sim \eta^{\lambda} \log L$ for large L and small η.

The fact that $\langle T\rangle$ and $\langle T^2\rangle^{1/2}$ are both proportional to $\log L$ for fixed η, and both proportional to η^{λ} for fixed L suggests that the distribution of reversal times may have the scaling form $p(T) \sim f(T/\langle T\rangle)/\langle T\rangle$ asymptotically for large L and small η. Figure 3(b) shows the scaled distribution for various values of L and η—the data collapse is fairly good.

We now present a simple argument which may explain why a condensate containing N particles flips direction in a time of order $\log N$ (or equivalently $\log L$ if the density is fixed) for large N. Consider a system in which, at $t = 0$, all N particles are positioned at $x = 0$ and have velocity $+1$. Let $q(x,t)$ be the probability that a given particle in this initial flock has position x at time t. In the limit $\eta \to 0$, each particle hops forward with probability $1/N$ in a time Δt so that in the limit $N \to \infty$, $\Delta t = 1/N$, the distribution $q(x,t)$ is Poisson:

$$q(x,t) = \frac{e^{-t}\, t^x}{x!}. \tag{2}$$

For η nonzero, this distribution will be modified by the flipping of particle velocities. For example, particles will leave the back of the condensate when η is nonzero but here we neglect this effect.

We now ask the following question: if some perturbation or fluctuation (due to collision with a small flock or the spontaneous flipping of particles at the front of the condensate) should cause all the particles in the condensate having position $x > z$ to flip velocity, will this in turn cause the particles with position $x = z$ to flip? We make the approximation that the particles with $x > z$ form a 'shock' moving back through the condensate so that at time t they all have position $z+1$ and velocity -1. Then the particles with position z will flip if $R(z,t) > 1$, where

$$R(z,t) = \frac{1}{q(z-1,t)+q(z,t)} \sum_{x=z+1}^{\infty} q(x,t) = \frac{t}{z+t} \sum_{x=1}^{\infty} \frac{z!\, t^x}{(z+x)!}. \tag{3}$$

For a given position z we define a critical time $t_c(z)$ through the solution of $R(z,t_c) = 1$ so that for $t > t_c(z)$, the particles at position z are susceptible to flipping.

It is easy to see that $R(z,t+1) > R(z+1,t)$ so that if the particles at $z+1$ are susceptible to flipping (i.e. if $R(z+1,t) > 1$) then those at z are subsequently also susceptible. Hence, the reversal time of the entire condensate is determined by the time one must wait for the leading particles to flip and it suffices to find $t_c(z)$ in the limit $z \to \infty$. It is straightforward to show that $R(z,t) \to s^2/(1-s^2)$ as $z \to \infty$ with $t = sz$; this implies that $t_c(z) \to z/\sqrt{2}$. Therefore, t_c is approximately proportional to z for large z.

For a condensate containing a large but finite number of particles N, we estimate $z^*(N,t)$, the position of the leading edge of the condensate, through the expression $q(z^*,t) = 1/N$. For the Poisson distribution when $t \propto z^*$ we find that $z^* \sim \log N$. Therefore our estimate for the time at which the condensate becomes susceptible to reversal is $t_c(z^*)$ which, being proportional to z^*, is of the order of $\log N$.

Since this simple analysis makes the assumption that one can decouple the spreading of the condensate and the fluctuations which may cause it to reverse its direction, it makes no predictions about the non-trivial η dependence of the reversal time and the scaling behaviour of the distribution of reversal times.

This completes our discussion of the condensed phase; we now turn to a numerical finite-size scaling analysis [8] of the phase transition between the condensed phase and the homogeneous phase for fixed $\rho = 1$. We have performed Monte Carlo simulations for system sizes between $L = 50$ and $L = 2000$, averaging over between 5×10^6 and 2×10^7 time steps in the steady state for each set of parameters. We have found that the absolute value of the mean particle velocity V, defined by

$$V = \frac{1}{N} \left| \sum_{\mu=1}^{N} v_\mu \right|, \tag{4}$$

is a convenient order parameter since, between reversals in the condensed phase, the majority of particles have the same velocity, that of the condensate. During the long (but $\mathcal{O}(\log L)$) time intervals between reversals, V fluctuates about some well-defined mean value. Figure 4(a) shows a plot of $\langle V \rangle$ against η for several different system sizes. The crossover from the disordered to the ordered regime becomes sharper with increasing system size, suggesting the presence of a continuous phase transition.

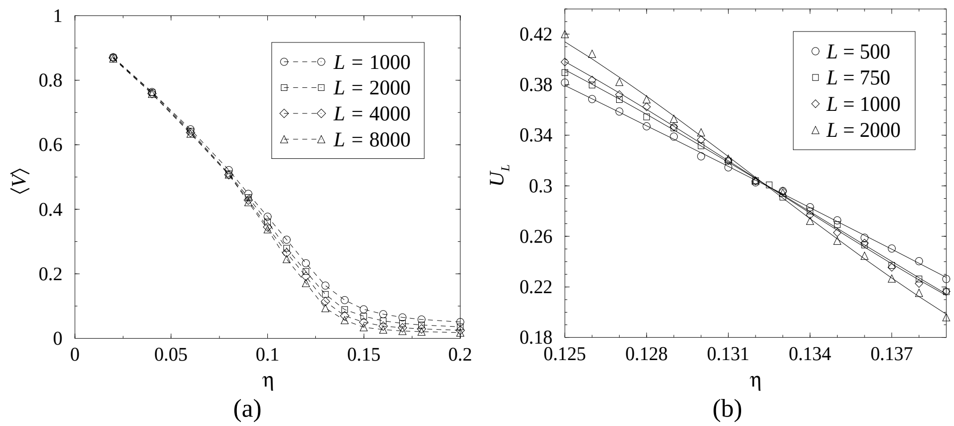


Figure 4. (a) $\langle V \rangle$ as a function of η for various system sizes with $\rho = 1$. (b) $U_L(\eta)$ for various system sizes in the vicinity of η_c, given by the common intersection point. The solid lines are cubic polynomial fits to the data.

In calculating critical quantities, it is useful to define both the 'static susceptibility' and the fourth-order cumulant [8], given respectively by

$$\chi(L) = L[\langle V^2 \rangle_L - \langle V \rangle_L^2] \;\; ; \qquad U_L = 1 - \frac{\langle V^4 \rangle_L}{3\langle V^2 \rangle_L}, \tag{5}$$

with the angle brackets indicating steady state time averages. In analogy with magnetic systems we assume that, near criticality, $\langle V \rangle \sim (\eta_c - \eta)^\beta$, $\chi \sim |\eta_c - \eta|^{-\gamma}$ and $\xi \sim |\eta_c - \eta|^{-\nu}$, where ξ is the correlation length.

Standard finite-size scaling theory [8] leads to the following relations at criticality:

$$\langle V \rangle_L \sim L^{-\beta/\nu}; \qquad \chi(L) \sim L^{\gamma/\nu}; \qquad U_L = U^*. \tag{6}$$

The critical point η_c can therefore be identified as the value of η at which U_L takes on its fixed point value U^*. Figure 4(b) shows that the curves $U_L(\eta)$ do indeed have a common intersection point at $\eta_c = 0.1325 \pm 0.001$. Also, we find $U^* = 0.30 \pm 0.01$.

The slopes of the functions $U_L(\eta)$ at $\eta = \eta_c$ can be used to calculate the critical exponent ν since [8]

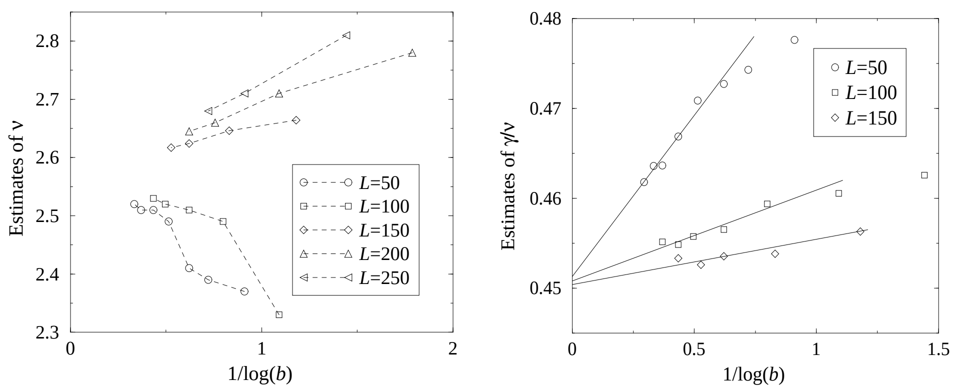
$$1/\nu = \log\left[(\partial U_{bL}/\partial U_L)|_{\eta_c} \right] / \log b \;\; + \;\; \text{corrections to scaling.} \tag{7}$$

We have found that $U_L(\eta)$ is approximately linear near η_c for L less than about 1000, allowing a reasonably precise estimate of the derivative in (7). In figure 5(a) we plot estimates of ν obtained using (7) against $1/\log b$ for various L. Extrapolating $b \to \infty$, thus taking account of corrections to scaling, we estimate $\nu = 2.57 \pm 0.05$.

The other independent exponent γ/ν is given by

$$\gamma/\nu = \log[\chi(bL, \eta_c)/\chi(L, \eta_c)]/\log b \;\; + \;\; \text{corrections to scaling.} \tag{8}$$

Figure 5(b) shows the resulting estimates of γ/ν plotted against $1/\log b$ for different values of L. As before, we extrapolate $b \to \infty$ and estimate $\gamma/\nu = 0.451 \pm 0.005$.



(a) (b)

Figure 5. Estimates of (a) ν and (b) γ/ν, obtained using (7) and (8), plotted against $1/\log b$ for various system sizes. The lines in both are guides to the eye.

The finite-size scaling formalism relies on the fact that the hyperscaling relation $d = \gamma/\nu + 2\beta/\nu$ holds [8]. At $\eta = \eta_c$ one can calculate β/ν via

$$\beta/\nu = -\log[\langle V \rangle_{bL}/\langle V \rangle_L]/\log b \quad + \quad \text{corrections to scaling.} \tag{9}$$

We were unable to estimate β/ν as precisely as γ/ν but, within error bounds, we have found that hyperscaling does indeed hold. Thus, our final estimates for the critical exponents are $\beta = 0.705 \pm 0.02$, $\gamma = 1.16 \pm 0.04$ and $\nu = 2.57 \pm 0.05$.

In reference [5] β was estimated (without finite-size scaling) for an off-lattice model of 1d flocking to be 0.6 ± 0.05. While our value of $\beta = 0.705 \pm 0.02$ is not in close agreement with this, it is still possible that both models are in the same universality class. Measurement of U^* in the off-lattice model would provide a good test of universality.

Above, we have used standard finite-size scaling to study the order parameter distribution. The success of this approach is surprising when one considers that the order parameter V does not fully reflect the alternating nature of the steady state in the condensed phase—the 'spreading' of the condensate is not captured.

We have argued that spontaneous symmetry breaking does not occur in our model and that the reversal times are $\mathcal{O}(\log L)$. This contrasts with equilibrium systems where reversal times exponentially large in L result from surmounting free energy barriers. It is difficult to see how $\mathcal{O}(\log L)$ reversal times could be explained in an analogous fashion; the reversal mechanism is fundamentally nonequilibrium.

Finally, we comment on the effects of imposing a restriction on the local particle density in our model; this is achieved by setting the maximum number of particles allowed on a site to be M. Not surprisingly, we have found that for any finite $M < N$, the condensed phase is suppressed; instead, for small η, the system forms 'domains' of maximally occupied sites. Each of these domains comprises two competing sub-domains composed of particles with velocities $+1$ and -1. The interface between the two sub-domains performs a random walk as a result of the flipping of particle velocities and the

dynamics resembles that of a 1d ferromagnet.

Acknowledgments

We thank M E Cates for many useful discussions and a critical reading of the manuscript and N B Wilding for helpful advice on finite-size scaling.

References

[1] Vicsek T, Czirók A, Ben-Jacob E, Cohen I and Shochet O 1995 *Phys. Rev. Lett.* **75** 1226
[2] Czirók A, Stanley H E and Vicsek T 1997 *J. Phys. A: Math. Gen.* **30** 1375
[3] Toner J and Tu Y 1998 *Phys. Rev.* E **58** 4828
[4] Bussemaker H J, Deutsch A and Geigant E 1997 *Phys. Rev. Lett.* **78** 5018
[5] Czirók A, Barabási A and Vicsek T 1997 Preprint cond-mat/9712154
[6] Albano E V 1996 *Phys. Rev. Lett.* **77** 2129
[7] Shimoyama N, Sugawara K, Mizuguchi T, Hayakawa Y and Sano M 1996 *Phys. Rev. Lett.* **76** 3870
[8] Binder K and Heermann D W 1997 *Monte Carlo Simulation in Statistical Physics*. Solid-State Sciences (Berlin Heidelberg: Springer), 3rd edn.